NAG Fortran Library Routine Document

F02FDF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

F02FDF computes all the eigenvalues, and optionally all the eigenvectors, of a real symmetric-definite generalized eigenproblem.

2 Specification

```
      SUBROUTINE F02FDF (ITYPE, JOB, UPLO, N, A, LDA, B, LDB, W, WORK, LWORK,
     1                   IFAIL)
      INTEGER            ITYPE, N, LDA, LDB, LWORK, IFAIL
      double precision   A(LDA,*), B(LDB,*), W(*), WORK(LWORK)
      CHARACTER*1        JOB, UPLO
```

3 Description

F02FDF computes all the eigenvalues, and optionally all the eigenvectors, of a real symmetric-definite generalized eigenproblem of one of the following types:

1. $Az = \lambda Bz$
2. $ABz = \lambda z$
3. $BAz = \lambda z$

Here A and B are symmetric, and B must be positive-definite.

The method involves implicitly inverting B; hence if B is ill-conditioned with respect to inversion, the results may be inaccurate (see Section 7).

Note that the matrix Z of eigenvectors is not orthogonal, but satisfies the following relationships for the three types of problem above:

1. $Z^{\mathrm{T}}BZ = I$
2. $Z^{\mathrm{T}}BZ = I$
3. $Z^{\mathrm{T}}B^{-1}Z = I$

4 References

Golub G H and Van Loan C F (1996) *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

Parlett B N (1998) *The Symmetric Eigenvalue Problem* SIAM, Philadelphia

5 Parameters

1: ITYPE – INTEGER *Input*

On entry: indicates the type of problem.

ITYPE = 1

The problem is $Az = \lambda Bz$;

ITYPE = 2

The problem is $ABz = \lambda z$;

ITYPE = 3

The problem is $BAz = \lambda z$.

Constraint: ITYPE = 1, 2 or 3.

2: JOB – CHARACTER*1 *Input*

On entry: indicates whether eigenvectors are to be computed.

JOB = 'N'

Only eigenvalues are computed.

JOB = 'V'

Eigenvalues and eigenvectors are computed.

Constraint: JOB = 'N' or 'V'.

3: UPLO – CHARACTER*1 *Input*

On entry: indicates whether the upper or lower triangular parts of A and B are stored.

UPLO = 'U'

The upper triangular parts of A and B are stored.

UPLO = 'L'

The lower triangular parts of A and B are stored.

Constraint: UPLO = 'U' or 'L'.

4: N – INTEGER *Input*

On entry: n, the order of the matrices A and B.

Constraint: N $\geq$ 0.

5: A(LDA,$*$) – ***double precision*** array *Input/Output*

Note: the second dimension of the array A must be at least max(1, N).

On entry: the n by n symmetric matrix A.

If UPLO = 'U', the upper triangle of A must be stored and the elements of the array below the diagonal need not be set.

If UPLO = 'L', the lower triangle of A must be stored and the elements of the array above the diagonal need not be set.

On exit: if JOB = 'V', A contains the matrix Z of eigenvectors, with the ith column holding the eigenvector z_i associated with the eigenvalue λ_i (stored in W(i)).

If UPLO = 'U', the upper triangular part of A is overwritten.

If UPLO = 'L', the lower triangular part of A if overwritten.

6: LDA – INTEGER *Input*

On entry: the first dimension of the array A as declared in the (sub)program from which F02FDF is called.

Constraint: LDA $\geq$ max(1, N).

7: B(LDB,$*$) – ***double precision*** array *Input/Output*

Note: the second dimension of the array B must be at least $\max(1, N)$.

On entry: the n by n symmetric positive-definite matrix B.

If UPLO = 'U', the upper triangle of B must be stored and the elements of the array below the diagonal are not referenced.

If UPLO = 'L', the lower triangle of B must be stored and the elements of the array above the diagonal are not referenced.

On exit: the upper or lower triangle of B (as specified by UPLO) is overwritten by the triangular factor U or L from the Cholesky factorization of B as $U^{T}U$ or LL^{T}.

8: LDB – INTEGER *Input*

On entry: the first dimension of the array B as declared in the (sub)program from which F02FDF is called.

Constraint: $LDB \geq \max(1, N)$.

9: W($*$) – ***double precision*** array *Output*

Note: the dimension of the array W must be at least $\max(1, N)$.

On exit: the eigenvalues in ascending order.

10: WORK(LWORK) – ***double precision*** array *Workspace*
11: LWORK – INTEGER *Input*

On entry: the dimension of the array WORK as declared in the (sub)program from which F02FDF is called. On some high-performance computers, increasing the dimension of WORK will enable the routine to run faster; a value of $64 \times N$ should allow near-optimal performance on almost all machines.

Constraint: $LWORK \geq \max(1, 3 \times N)$.

12: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this parameter you should refer to Chapter P01 for details.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this parameter the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

On entry, ITYPE $\neq$ 1, 2 or 3,
or JOB $\neq$ 'N' or 'V',
or UPLO $\neq$ 'U' or 'L',
or $N < 0$,
or $LDA < \max(1, N)$,

or LDB < max(1, N),
or LWORK < max(1, 3 × N).

IFAIL = 2

The QR algorithm failed to compute all the eigenvalues.

IFAIL = 3

The matrix B is not positive-definite.

7 Accuracy

If λ_i is an exact eigenvalue, and $\tilde{\lambda}_i$ is the corresponding computed value, then

for problems of the form $Az = \lambda Bz$,

$$\left|\tilde{\lambda}_i - \lambda_i\right| \leq c(n)\epsilon\|A\|_2\left\|B^{-1}\right\|_2;$$

for problems of the form $ABz = \lambda z$ or $BAz = \lambda z$,

$$\left|\tilde{\lambda}_i - \lambda_i\right| \leq c(n)\epsilon\|A\|_2\|B\|_2.$$

Here $c(n)$ is a modestly increasing function of n, and ϵ is the ***machine precision***.

If z_i is the corresponding exact eigenvector, and $\tilde{z}_i$ is the corresponding computed eigenvector, then the angle $\theta(\tilde{z}_i, z_i)$ between them is bounded as follows:

for problems of the form $Az = \lambda Bz$,

$$\theta(\tilde{z}_i, z_i) \leq \frac{c(n)\epsilon\|A\|_2\left\|B^{-1}\right\|_2(\kappa_2(B))^{1/2}}{\min_{i \neq j}\left|\lambda_i - \lambda_j\right|};$$

for problems of the form $ABz = \lambda z$ or $BAz = \lambda z$,

$$\theta(\tilde{z}_i, z_i) \leq \frac{c(n)\epsilon\|A\|_2\|B\|_2(\kappa_2(B))^{1/2}}{\min_{i \neq j}\left|\lambda_i - \lambda_j\right|}.$$

Here $\kappa_2(B)$ is the condition number of B with respect to inversion defined by: $\kappa_2(B) = \|B\|.\left\|B^{-1}\right\|$. Thus the accuracy of a computed eigenvector depends on the gap between its eigenvalue and all the other eigenvalues, and also on the condition of B.

8 Further Comments

F02FDF calls routines from LAPACK in Chapter F08. It first reduces the problem to an equivalent standard eigenproblem $Cy = \lambda y$. It then reduces C to tridiagonal form T, using an orthogonal similarity transformation: $C = QTQ^{\mathrm{T}}$. To compute eigenvalues only, the routine uses a root-free variant of the symmetric tridiagonal QR algorithm to reduce T to a diagonal matrix Λ. If eigenvectors are required, the routine first forms the orthogonal matrix Q that was used in the reduction to tridiagonal form; it then uses the symmetric tridiagonal QR algorithm to reduce T to Λ, using a further orthogonal transformation: $T = S\Lambda S^{\mathrm{T}}$; and at the same time accumulates the matrix $Y = QS$, which is the matrix of eigenvectors of C. Finally it transforms the eigenvectors of C back to those of the original generalized problem.

Each eigenvector z is normalized so that:

for problems of the form $Az = \lambda Bz$ or $ABz = \lambda z$, $z^{\mathrm{T}}Bz = 1$;

for problems of the form $BAz = \lambda z$, $z^{\mathrm{T}}B^{-1}z = 1$.

The time taken by the routine is approximately proportional to n^3.

9 Example

To compute all the eigenvalues and eigenvectors of the problem $Az = \lambda Bz$, where

$$A = \begin{pmatrix} 0.24 & 0.39 & 0.42 & -0.16 \\ 0.39 & -0.11 & 0.79 & 0.63 \\ 0.42 & 0.79 & -0.25 & 0.48 \\ -0.16 & 0.63 & 0.48 & -0.03 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4.16 & -3.12 & 0.56 & -0.10 \\ -3.12 & 5.03 & -0.83 & 1.09 \\ 0.56 & -0.83 & 0.76 & 0.34 \\ -0.10 & 1.09 & 0.34 & 1.18 \end{pmatrix}.$$

9.1 Program Text

```
*     F02FDF Example Program Text
*     Mark 16 Release. NAG Copyright 1992.
*     .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
      INTEGER          NMAX, LDA, LDB, LWORK
      PARAMETER        (NMAX=8,LDA=NMAX,LDB=NMAX,LWORK=64*NMAX)
*     .. Local Scalars ..
      INTEGER          I, IFAIL, ITYPE, J, N
      CHARACTER        UPLO
*     .. Local Arrays ..
      DOUBLE PRECISION A(LDA,NMAX), B(LDB,NMAX), W(NMAX), WORK(LWORK)
*     .. External Subroutines ..
      EXTERNAL         F02FDF, X04CAF
*     .. Executable Statements ..
      WRITE (NOUT,*) 'F02FDF Example Program Results'
*     Skip heading in data file
      READ (NIN,*)
      READ (NIN,*) N
      IF (N.LE.NMAX) THEN
*
*        Read A and B from data file
*
         READ (NIN,*) UPLO
         IF (UPLO.EQ.'U') THEN
            READ (NIN,*) ((A(I,J),J=I,N),I=1,N)
            READ (NIN,*) ((B(I,J),J=I,N),I=1,N)
         ELSE IF (UPLO.EQ.'L') THEN
            READ (NIN,*) ((A(I,J),J=1,I),I=1,N)
            READ (NIN,*) ((B(I,J),J=1,I),I=1,N)
         END IF
*
*        Compute eigenvalues and eigenvectors
*
         ITYPE = 1
         IFAIL = 0
*
         CALL F02FDF(ITYPE,'Vectors',UPLO,N,A,LDA,B,LDB,W,WORK,LWORK,
     +               IFAIL)
*
         WRITE (NOUT,*)
         WRITE (NOUT,*) 'Eigenvalues'
         WRITE (NOUT,99999) (W(I),I=1,N)
         WRITE (NOUT,*)
*
         CALL X04CAF('General',' ',N,N,A,LDA,'Eigenvectors',IFAIL)
*
      END IF
      STOP
*
99999 FORMAT (3X,(8F11.4))
      END
```

9.2 Program Data

```
F02FDF Example Program Data
  4                            :Value of N
  'L'                          :Value of UPLO
  0.24
  0.39  -0.11
  0.42   0.79  -0.25
 -0.16   0.63   0.48  -0.03   :End of matrix A
  4.16
 -3.12   5.03
  0.56  -0.83   0.76
 -0.10   1.09   0.34   1.18   :End of matrix B
```

9.3 Program Results

```
 F02FDF Example Program Results

 Eigenvalues
       -2.2254    -0.4548     0.1001     1.1270

 Eigenvectors
             1          2          3          4
 1     -0.0690    -0.3080     0.4469     0.5528
 2     -0.5740    -0.5329     0.0371     0.6766
 3     -1.5428     0.3496    -0.0505     0.9276
 4      1.4004     0.6211    -0.4743    -0.2510
```